ALGEBRAIC CURVES
SOLUTIONS SHEET 6

Unless otherwise specified, k is an algebraically closed field.

Exercise 1. Let V, W be varieties and assume that W is affine.

(1) Show that there is a bijection $Hom_{Var}(V, W) \simeq Hom_{k-alg}(\mathcal{O}(W), \mathcal{O}(V))$.
(2) Show that there is a bijection $\mathcal{O}(V) \simeq Hom_{Var}(V, \mathbb{A}^1_k)$.
(3) Show that $\mathcal{O}(\mathbb{P}^1_k) \simeq k$.
(4) Show that $\mathcal{O}(\mathbb{P}^n_k) \simeq k$ for all $n \geq 1$.

Solution 1.

(1) Let $\phi \colon V \to W$ be a morphism of varieties. Then $\phi^* \colon \mathcal{O}(W) \to \mathcal{O}(V)$ defined by $f \mapsto f \circ \phi$ is a $k-$algebra morphism.

To go in the other direction, let $\Phi \colon \mathcal{O}(W) \to \mathcal{O}(V)$ be a k-algebra homomorphism. There are at least two approaches:

Approach 1: We use that by Exercise 3.5, points of W correspond to maximal ideals of $\Gamma(U)$, and as U is affine we have $\Gamma(U) = \mathcal{O}(U)$. Now notice that for any $P \in V$, we have an evaluation morphism $\mathrm{ev}_P \colon \mathcal{O}(V) \to k$, defined by $f \mapsto f(P)$. Note that this is a morphism of k-algebras. Therefore, we obtain a k-algebra homomorphism $\mathrm{ev}_P \circ \Phi \colon \mathcal{O}(W) \to k$, so in particular it is surjective (on elements of k it is the identity). Hence $\ker(\mathrm{ev}_P \circ \Phi)$ is a maximal ideal of $\mathcal{O}(W)$, and hence it correspond to a point $Q_P \in W$. Let us define a map $\Phi^\flat \colon V \to W$ by $\Phi^\flat(P) := Q_P$. Let us show that $\Phi^\flat$ is a morphism of varieties.
First we have to show continuity. So let $W' \subseteq W$ be closed, and let $I(W') \subseteq \mathcal{O}(W)$ be its ideal. Note that as the correspondence in Exercise 3.5 is inclusion reversing, we have for all $P \in V$

$$\begin{aligned} \Phi^\flat(P) \in W' &\iff V(I(\Phi^\flat(P))) \subseteq V(I(W')) \\ &\iff I(\Phi^\flat(P)) \supseteq I(W') \\ &\iff \ker(\mathrm{ev}_P \circ \Phi) \supseteq I(W') \\ &\iff \forall f \in I(W') : \ \Phi(f)(P) = 0 \\ &\iff P \in \bigcap_{f \in I(W')} \Phi(f)^{-1}(0) \end{aligned}$$

where we used that by definition the ideal of $\{\Phi^\flat(P)\}$ is $\ker(\mathrm{ev}_P \circ \Phi)$. As for any f, $\Phi(f) \in \mathcal{O}(V)$ is continuous as a function $V \to k$, we arrived in the end at an intersection of closed subsets in the end. Hence

we obtain that $(\Phi^\flat)^{-1}(W') = \bigcap_{f \in I(W')} \Phi(f)^{-1}(\{0\})$ is closed, so $\Phi^\flat$ is continuous.
Now to verify that it is indeed a morphism, let us first show that $\Phi(f) = f \circ \Phi^\flat$ for any element $f \in \mathcal{O}(W)$. This comes from the following very general observation: if X is any affine variety, $Q \in X$ is any point and $g \in \mathcal{O}(X)$ any function, then $g - g(Q) \in I(Q)$. Therefore, in our present situation we have

$$f - f(\Phi^\flat(P)) \in I(\Phi^\flat(P)) = \ker(\mathrm{ev}_p \circ \Phi)$$

and thus

$$0 = \mathrm{ev}_p \circ \Phi \left(f - f(\Phi^\flat(P))\right) = \Phi(f)(P) - f(\Phi^\flat(P)).$$

As this holds for every $P \in V$, we indeed conclude that $\Phi(f) = f \circ \Phi^\flat$.
Finally, let us show that $\Phi^\flat$ is a morphism by using Lemma 3.8. As W is affine, we can view it as a Zariski closed subset $W \subseteq \mathbb{A}^n$, and let $x_1, \dots, x_n \colon \mathbb{A}^n \to \mathbb{A}^1$ be the coordinates. By abuse of notation, we also denote by $x_1, \dots, x_n$ their images in $\mathcal{O}(W)$ (i.e. their restrictions to W). Then we have for all i that

$$x_i \circ \Phi^\flat = \Phi(x_i) \in \mathcal{O}(V),$$

i.e. $x_i \circ \Phi^\flat$ is regular. By Lemma 3.8, we conclude that $\Phi^\flat$ is a morphism of varieties.
We are left to show that the two constructions are mutually inverse.
On the one hand, let $\phi \colon V \to W$ be any morphism of varieties, and let $P \in V$ be arbitrary. Then for any $f \in \mathcal{O}(W)$ we have

$$f(\phi^{*\flat}(P)) = \phi^*(f)(P) = f(\phi(P)).$$

In particular, by taking $f = x_i$ for $1 \leq i \leq n$, we obtain that $\phi^{*\flat}(P) = \phi(P)$, and thus $\phi^{*\flat} = \phi$.
On the other hand, let $\Phi \colon \mathcal{O}(W) \to \mathcal{O}(V)$ be any morphism of k-algebras, and let $f \in \mathcal{O}(W)$ be arbitrary. Then for any $P \in V$, we have

$$\Phi^{\flat *}(f)(P) = f \circ \Phi^\flat(P) = \Phi(f)(P),$$

so $\Phi^{\flat *}(f) = \Phi(f)$, and as f was arbitrary we conclude $\Phi^{\flat *} = \Phi$.

Approach 2: Another way of defining $\Phi^\flat$ is the following: fix a closed embedding $W \subseteq \mathbb{A}^n$, and denote by $x_1, \dots, x_n$ the coordinates. Denote by $\overline{x_1}, \dots, \overline{x_n} \in \mathcal{O}(W)$ their restriction to W. We define a map

$$\begin{aligned} \Phi^\flat \colon V &\to W \\ P &\mapsto (\Phi(\overline{x_1})(P), \dots, \Phi(\overline{x_n})(P)). \end{aligned}$$

Firstly, we have to show that this is well-defined, i.e. that for all $P \in V$ we indeed have $(\Phi(x_1)(P), \dots, \Phi(x_n)(P)) \in W$ (a priori it is just some point in $\mathbb{A}^n$). To do so, let $f \in I(W) \subseteq k[x_1, \dots, x_n]$ be arbitrary. As f is a polynomial and polynomials commute with morphisms of rings, we have

$$\begin{aligned} f(\Phi(\overline{x_1})(P), \dots, \Phi(\overline{x_n})(P)) &= f(\mathrm{ev}_P \circ \Phi(\overline{x_1}), \dots, \mathrm{ev}_P \circ \Phi(\overline{x_n})) \\ &= \mathrm{ev}_P \circ \Phi(\underbrace{f(\overline{x_1}, \dots, \overline{x_n})}_{=0}) \\ &= 0, \end{aligned}$$

where in the last step we used that $f \in I(W)$. Hence we indeed conclude that $\Phi^\flat(P) \in W$ for all $P \in W$.
To verify that $\Phi^\flat$ is a morphism, we again use Lemma 3.8: for every $1 \leq i \leq n$ we have by construction that $x_i \circ \Phi^\flat = \Phi(\overline{x_i}) \in \mathcal{O}(V)$, i.e. it is regular on V. Hence $\Phi^\flat$ is indeed a morphism.
Finally, we have to show that the two constructions are mutually inverse. On the one hand, let $\phi \colon V \to W$ be any morphis of varieties, and let $P \in V$ be arbitrary. Then

$$\begin{aligned} \phi^{*\flat}(P) &= (\phi^*(\overline{x_1})(P), \dots, \phi^*(\overline{x_n})(P)) \\ &= (\overline{x_1}(\phi(P)), \dots, \overline{x_n}(\phi(P))) \\ &= (\phi(P)_1, \dots, \phi(P)_n) \\ &= \phi(P), \end{aligned}$$

so we conclude $\phi^{*\flat} = \phi$.
On the other hand, let $\Phi \colon \mathcal{O}(W) \to \mathcal{O}(V)$ be arbitrary, and let also $\overline{f} = f + I(W) \in \mathcal{O}(W)$ and $P \in V$ be arbitrary. Then we have

$$\begin{aligned} \Phi^{\flat *}(\overline{f})(P) &= \overline{f}(\Phi^\flat(P)) \\ &= \overline{f}(\Phi(\overline{x_1})(P), \dots, \Phi(\overline{x_n})(P)) \\ &= f(\Phi(\overline{x_1})(P), \dots, \Phi(\overline{x_n})(P)) \\ &= \Phi(\underbrace{f(\overline{x_1}, \dots, \overline{x_n})}_{=\overline{f}})(P) \\ &= \Phi(\overline{f})(P), \end{aligned}$$

where we again used that polynomials commute with ring morphisms, so we conclude $\Phi^{\flat *} = \Phi$.

(2) Clearly from (1) we get that $\mathrm{Hom}_{Var}(V, \mathbb{A}^1) \simeq \mathrm{Hom}_{k-alg}(k[x], \mathcal{O}(V))$. So it suffices to see that $\mathrm{Hom}_{k-alg}(k[x], \mathcal{O}(V)) \simeq \mathcal{O}(V)$. This is clear, since a k–algebra homomorphism from $k[x]$ into any k–algebra is completely determined by choosing the image of x. More concretely, one can verify

that

$$\begin{aligned}\mathrm{Hom}_{k-alg}(k[x], \mathcal{O}(V)) &\to \mathcal{O}(V)\\ \Phi &\mapsto \Phi(x)\end{aligned}$$

is a bijection, where the preimage of $f \in \mathcal{O}(V)$ is the unique k–algebra homomorphism $k[x] \to \mathcal{O}(V)$ sending x to f.

(3) Using question 2, $\mathcal{O}(\mathbb{P}^1_k) = Hom_{Var}(\mathbb{P}^1, \mathbb{A}^1)$. Call x_1 and x_2 the projective corrdinates and $\mathbb{P}^1 = U_1 \cup U_2$ the corresponding standard affine open cover, with isomorphisms $\varphi_i \colon \mathbb{A}^1 \to U_i$. Let $f \colon \mathbb{P}^1 \to \mathbb{A}^1$ be arbitrary. Then we obtain morphisms $f \circ \varphi_i \colon \mathbb{A}^1 \to \mathbb{A}^1$, which are given by some polynomial $p_i \in k[x]$. Note that for any $a \in \mathbb{A}^1 \setminus \{0\}$ we have

$$p_2(a) = f \circ \varphi_2(a) = f([a : 1]) = f([1 : 1/a]) = f \circ \varphi_1(1/a) = p_1(1/a).$$

As this equality holds on the dense open set $\mathbb{A}^1 \setminus \{0\}$, we conclude that $p_2(x)$ and $p_1(1/x)$ agree as elements of the localization $k[x, x^{-1}]$. But as both p_1 and p_2 are polynomials, this can only happen if both of them are constants. Hence f has to be constant as well. Therefore, the map $k \to \mathcal{O}(\mathbb{P}^1)$ sending an element a of k to the function constantly equal to k is a bijection.

A nice notational trick is the following: let for $\varphi_1 \colon \mathbb{A}^1 \to U_1$ we denote the coordinate of $\mathbb{A}^1$ by the symbol x_2/x_1, and for $\varphi_2 \colon \mathbb{A}^1 \to U_2$ we denote the coordinate of $\mathbb{A}^1$ by x_1/x_2.

Because of the way φ_1, φ_2 are defined, two functions $p_1 \colon \mathbb{A}^1_{x_2/x_1} \to \mathbb{A}^1$ resp. $p_2 \colon \mathbb{A}^1_{x_1/x_2} \to \mathbb{A}^1$ glue to a function $f \colon \mathbb{P}^1 \to \mathbb{A}^1$ (i.e. $p_1 \circ \varphi_1^{-1}$ and $p_2 \circ \varphi_2^{-1}$ agree on the intersection $U_1 \cap U_2$) if and only if $p_1(x_2/x_1) = p_2(x_1/x_2)$ as elements of $k[x_1/x_2, x_2/x_1]$. One could also solve the exercise this way.

(4) There are at least three approaches to solve this

Approach 1: We can mimic the solution to point (3), but first we need to introduce some notation, and discuss how to think of regular funcitons on $\mathbb{P}^n$. Denote by $x_1, \ldots, x_{n+1}$ the projective coordinates on $\mathbb{P}^n$, by $\mathbb{P}^n = U_1 \cup \cdots \cup U_{n+1}$ the standard open cover, with isomorphisms $\varphi_i \colon \mathbb{A}^n \to U_i$. Denote the coordinates of $\mathbb{A}^n$ by $y_1, \ldots, y_n$. Then in particular, we have an isomorphism $\widetilde{\varphi}_i \colon \mathcal{O}_{\mathbb{P}^n}(U_i) \to k[y_1, \ldots, y_n]$. So which regular functions on U_i are mapped to the variables $y_1, \ldots, y_n$? Note that the functions $x_1/x_i, \ldots x_{i-1}/x_i, x_{i+1}/x_i, \ldots, x_{n+1}/x_i$ on $\mathbb{P}^n$ are elements of $\mathcal{O}_{\mathbb{P}^n}(U_i)$, and from the definition of φ_i, you can convince yourself that we have in fact $\widetilde{\varphi}_i(x_j/x_i) = y_j$ for $j < i$ and $\widetilde{\varphi}_i(x_j/x_i) = y_{j-1}$ for $j > i$. Therefore, $\mathcal{O}_{\mathbb{P}^n}(U_i)$ is generated as a k–algebra by $x_1/x_i, \ldots x_{i-1}/x_i, x_{i+1}/x_i, \ldots, x_{n+1}/x_i$, and these functions are algebraically independent over k. We may summarize this by writing

$$\mathcal{O}_{\mathbb{P}^n}(U_i) = k\left[\frac{x_1}{x_i}, \dots \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_{n+1}}{x_i}\right],$$

remembering that the x_j/x_i behave like the indeterminate variables in a polynomial ring.

Now by point (2) of Exercise 2 applied to $U_i \setminus V(x_j/x_i)$, we actually have

$$\mathcal{O}_{\mathbb{P}^n}(U_i \cap U_j) = k\left[\frac{x_1}{x_i}, \dots \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_{n+1}}{x_i}, \frac{x_i}{x_j}\right]$$

$$= \left\{\frac{p}{x_i^a x_j^b} \mid a, b \geq 0,\ p \in k[x_1, \dots, x_{n+1}]_{a+b},\ p \text{ not divisible by } x_i, x_j\right\}$$

Note that the image of the restriction map $\mathcal{O}_{\mathbb{P}^n}(U_i) \to \mathcal{O}_{\mathbb{P}^n}(U_i \cap U_j)$ is precisely the set of those elements which in the above description have $b = 0$. Similarly, the image of the restriction map $\mathcal{O}_{\mathbb{P}^n}(U_j) \to \mathcal{O}_{\mathbb{P}^n}(U_i \cap U_j)$ is the set of elements with $a = 0$. Therefore, the intersection of the images of $\mathcal{O}_{\mathbb{P}^n}(U_i)$ and $\mathcal{O}_{\mathbb{P}^n}(U_j)$ inside $\mathcal{O}_{\mathbb{P}^n}(U_i \cap U_j)$ is the set of elements with $a = b = 0$, i.e. it is just k.

Now to the actual problem: let $f \in \mathcal{O}(\mathbb{P}^n)$ be arbitrary. Then $f|_{U_i \cap U_j} \in \mathcal{O}_{\mathbb{P}^n}(U_i \cap U_j)$ lies in the images of both $\mathcal{O}_{\mathbb{P}^n}(U_i) \to \mathcal{O}_{\mathbb{P}^n}(U_i \cap U_j)$ and $\mathcal{O}_{\mathbb{P}^n}(U_j) \to \mathcal{O}_{\mathbb{P}^n}(U_i \cap U_j)$ (with preimages $f|_{U_i}$ resp. $f|_{U_j}$). By the above discussion, we must have $f|_{U_i \cap U_j} = c$ for some $c \in k$. But then the vanishing locus of the function $f - c \in \mathcal{O}(\mathbb{P}^n)$ contains the dense open set $U_i \cap U_j$, so we obtain that $V(f - c) = \mathbb{P}^n$, i.e. $f = c$.

Remark. This proof actually shows that for all $i \neq j$ we have $\mathcal{O}_{\mathbb{P}^n}(U_i \cup U_j) = k$. This is actually a stronger statement than $\mathcal{O}_{\mathbb{P}^n}(\mathbb{P}^n) = k$: indeed, as two regular functions which agree on a dense open subset agree globally, the restriction map $\mathcal{O}_{\mathbb{P}^n}(\mathbb{P}^n) \to \mathcal{O}_{\mathbb{P}^n}(U_i \cup U_j)$ is injective, so the latter being k implies that the former is as well.

On the other hand, it is not a coincidence that in our situation the map $\mathcal{O}_{\mathbb{P}^n}(\mathbb{P}^n) \to \mathcal{O}_{\mathbb{P}^n}(U_i \cup U_j)$ is an isomorphism. Indeed, there is a result known as *Hartogs Lemma*, which says that for a normal (google it!) variety X and an open set $U \subseteq X$ such that $\dim(X \setminus U) \leq \dim X - 2$, the restriction map $\mathcal{O}_X(X) \to \mathcal{O}_X(U)$ is an isomorphism. Note that $\mathbb{P}^n \setminus (U_i \cup U_j) \cong \mathbb{P}^{n-2}$, so taking for granted that $\mathbb{P}^n$ is normal, we see that $\mathcal{O}_{\mathbb{P}^n}(\mathbb{P}^n) \to \mathcal{O}_{\mathbb{P}^n}(U_i \cup U_j)$ is an isomorphism.

Approach 2: The function field $K(\mathbb{P}^n)$ of $\mathbb{P}^n$ is the following subfield of $k(x_1, \dots, x_{n+1})$:

$$K(\mathbb{P}^n) = \left\{\frac{p}{q} \mid f, g \in k[x_1, \dots, x_{n+1}] \text{ homog. of the same deg.}\right\} \subseteq k(x_1, \dots, x_{n+1}).$$

Note that the natural map $\mathcal{O}_{\mathbb{P}^n}(\mathbb{P}^n) \to K(\mathbb{P}^n)$ is injective, because if two regular functions agree on a dense open subset then they agree globally.

Now let $f \in \mathcal{O}_{\mathbb{P}^n}(\mathbb{P}^n)$ be arbitrary, and regard it as an element of $K(\mathbb{P}^n)$, where we can write $f = p/q$ with $p, q \in k[x_1, \ldots, x_{n+1}]$ homogeneous of the same degree. Now as $k[x_1, \ldots, x_{n+1}]$ is a UFD and irreducible factors of homogeneous polynomials are homogeneous, we may assume that p, q are coprime. In particular, if $f = a/b$ for a, b homogeneous of the same degree, we must have $p \mid a$ and $q \mid b$. So f is defined precisely on $\mathbb{P}^n \setminus V(q)$. But as $f \in \mathcal{O}_{\mathbb{P}^n}(\mathbb{P}^n)$ by assumption, we assume $V(q) = \emptyset$. This can only be the case if $q \in k$, and thus also $p \in k$. Hence $f = p/q$ inside $K(\mathbb{P}^n)$, and as $\mathcal{O}_{\mathbb{P}^n}(\mathbb{P}^n) \to K(\mathbb{P}^n)$ is injective, we conclude that f is constant.

Approach 3: Let $f \in \mathcal{O}_{\mathbb{P}^n}(\mathbb{P}^n)$ be arbitrary and consider it as a morphism $f\colon \mathbb{P}^n \to \mathbb{A}^1$ by point (2). Let $P, Q \in \mathbb{P}^n$ be arbitrary distinct points, and write $P = [p]$ resp. $Q = [q]$ for some $p, q \in \mathbb{A}^{n+1} \setminus \{0\}$. Now note that we have a morphism φ defined by the formula

$$\begin{aligned} \varphi\colon \mathbb{P}^1 &\to \mathbb{P}^n \\ [a : b] &\mapsto [ap + bq]. \end{aligned}$$

You can either verify directly that this is indeed a morphism, or use that it is a composition of inclusions of 'hyperplanes' at infinity $\mathbb{P}^m \hookrightarrow \mathbb{P}^{m+1}$ (sending $[p]$ to $[p : 0]$) and a projective change of coordinates, all of which we know to be morphisms.

Now note that $f \circ \varphi\colon \mathbb{P}^1 \to \mathbb{A}^1$, so by point (3) it follows that $f \circ \varphi$ is constant. In particular, we have

$$f(P) = f \circ \varphi([1 : 0]) = f \circ \varphi([0 : 1]) = f(Q).$$

As $P, Q \in \mathbb{P}^n$ were arbitrary, we conclude that f is constant.

Exercise 2. Let $n \geq 1$ and $f \in k[x_1, \ldots, x_n]$.

(1) Show that $\mathbb{A}^n_k - V(f)$ is affine. What is its ring of regular functions?
(2) Show that $\mathbb{A}^2_k - \{(0,0)\}$ is not affine. (Hint: compute the ring of regular functions).

Solution 2.

(1) Let $D(f)$ denote $\mathbb{A}^n \setminus V(f)$. Denote by $x_1, \ldots, x_n$ the coordinates of $\mathbb{A}^n$, and by $x_1, \ldots, x_n, y$ the coordinates of $\mathbb{A}^{n+1}$. On the one hand, consider the map

$$\begin{aligned} \varphi\colon D(f) &\to V(1 - yf) \subseteq \mathbb{A}^{n+1} \\ (x_1, \ldots, x_n) &\mapsto \left(x_1, \ldots, x_n, \frac{1}{f(x_1, \ldots, x_n)} \right); \end{aligned}$$

it is straightforward to see that the image of φ indeed lands in $V(1-yf)$. Note that $x_i \circ \varphi = x_i$ and $y \circ \varphi = 1/f$, all of which are regular on $D(f)$. Hence by Lemma 3.8 we conclude that φ is a morphism.

On the other hand, let $\pi\colon \mathbb{A}^{n+1} \to \mathbb{A}^n$ be the projection onto the first n-coordinates, which is a morphism. Then $\pi|_{V(1-yf)}\colon V(1-yf) \to \mathbb{A}^n$ is a morphism as well. Furthermore, you can verify that the image of $\pi|_{V(1-yf)}$ lands inside $D(f)$. As corestricting to an open subset preserves morphisms (this comes just from the fact that for inclusions of open subset $U \subseteq V \subseteq W$ we have $\mathcal{O}_V(U) = \mathcal{O}_W(U)$), we obtain that $\psi := \pi|_{V(1-yf)}^{D(f)}\colon V(1-yf) \to D(f)$ is a morphism.

Finally, it is straightforward to see that $\varphi \circ \psi = \mathrm{Id}_{V(1-yf)}$ and $\psi \circ \varphi = \mathrm{Id}_{D(f)}$, so they are isomorphisms, and in particular $D(f)$ is isomorphic to the affine variety $V(1-yf)$.

(2) Denote $W := \mathbb{A}^2 \setminus \{(0,0)\}$, and by x, y the coordinates on $\mathbb{A}^2$. We start by computing $\mathcal{O}(W)$: let $f \in \mathcal{O}(W)$ be arbitrary. There are two approaches to solving this

Approach 1: As $\mathbb{A}^2$ is irreducible, for any open subset $U \subseteq \mathbb{A}^2$ the natural map $\mathcal{O}(U) \to K(\mathbb{A}^2)$ from the regular functions on U into the function field of $\mathbb{A}^2$ is an injection. Note that $K(\mathbb{A}^2) = k(x,y)$, so we can write $f = a/b$ for some $a, b \in k[x,y]$ which are coprime. Assume by contradiction that b has positive degree. Note that then $V(b)$ contains infinitely many elements; we will prove this at the end. In particular, the set $V(b) \setminus \{(0,0)\}$ is non-empty, i.e. it contains some point p. But then f is not defined at p: if $c, d \in k[x,y]$ are such that $f = c/d$, we must have in particular that $b \mid d$, so also $p \in V(d)$. On the other hand, as $f \in \mathcal{O}(W)$ it must be defined at p, contradiction. Hence we must have $b \in k^\times$, and thus $f \in k[x,y]$. Hence we conclude that $\mathcal{O}(W) = k[x,y]$, i.e. the restriction map $\mathcal{O}(\mathbb{A}^2) \to \mathcal{O}(W)$ is an isomorphism.

We finish by proving that for non-constant $b \in k[x,y]$, the vanishing locus $V(b)$ is infinite: indeed, up to interchanging x and y, we may assume that a positive power of x appears in b. Then, write $b = b_0(y) + b_1(y)x + \cdots + b_d(y)x^d$ for some $b_0, \ldots, b_d \in k[y]$, with $b_d \neq 0$ and $d > 0$. In particular, $V(b_1, \ldots, b_d)$ is finite and thus $\mathbb{A}^1 \setminus V(b_1, \ldots, b_d)$ is infinite. Furthermore, for $c \in \mathbb{A}^1 \setminus V(b_1, \ldots, b_d)$, the polynomial $b(x,c) \in k[x]$ has positive degree, so as k is algebraically closed, there exists $x_c \in k$ with $b(x_c, c) = 0$. Hence $V(b)$ contains at least as many elements as $\mathbb{A}^1 \setminus V(b_1, \ldots, b_d)$, and thus is infinite.

Approach 2: Let $f \in \mathcal{O}(W)$ be arbitrary. Note that $W = D(x) \cup D(y)$, and by point (1) we have

$$\mathcal{O}(D(x)) = k[x, y, x^{-1}]$$
$$\mathcal{O}(D(y)) = k[x, y, y^{-1}]$$
$$\mathcal{O}(D(x) \cap D(y)) = \mathcal{O}(D(xy)) = k[x, y, (xy)^{-1}] = k[x, y, x^{-1}, y^{-1}].$$

It is then clear that the image of the restriction map $\mathcal{O}(D(x)) \to \mathcal{O}(D(xy))$ is precisely the set of those elements of $k[x, y, x^{-1}, y^{-1}]$ where no negative power of y appears, and the image of the restriction map $\mathcal{O}(D(y)) \to \mathcal{O}(D(xy))$ is precisely the set of those elments where no negative power of x appears. Hence, the intersection of these images is the set of those elements where no negative power of neither x nor y appear, i.e. it is $k[x, y] \subseteq k[x, y, x^{-1}, y^{-1}]$. As $f|_{D(xy)}$ lies in the intersection of these images, we conclude that $f|_{D(xy)} \in k[x, y]$, and as $D(xy)$ is dense open in W we conclude that $f \in k[x, y]$. In other words, the restriction map $\mathcal{O}(\mathbb{A}^2) \to \mathcal{O}(W)$ is an isomorphism.

Assume now by contradiction that $\mathcal{O}(W)$ is affine. Then by point (1) of Exercise 1, the restriction map $\Phi\colon \mathcal{O}(\mathbb{A}^2) \to \mathcal{O}(W)$ and its inverse $\Psi\colon \mathcal{O}(W) \to \mathcal{O}(\mathbb{A}^2)$ induce mutually inverse morphisms $\varphi\colon W \to \mathbb{A}^2$ and $\psi\colon \mathbb{A}^2 \to W$. If $\iota\colon W \hookrightarrow \mathbb{A}^2$ denotes the inclusion, then clearly $\Phi = - \circ \iota$, so in fact we see that $\varphi = \iota$. But then ψ has to be an inverse to the inclusion map, which doesn't exist (where is $(0, 0)$ sent?). We arrived at a contradiction, so W cannot be affine.

Remark. As in point (4) of Exercise 1, if we admit that $\mathbb{A}^2$ is normal, then by Hartogs Lemma it would follow immediately that $\mathcal{O}(W) = \mathcal{O}(\mathbb{A}^2)$, as $\mathbb{A}^2 \setminus W = \{(0, 0)\}$ has codimension 2.

Exercise 3. Let $\varphi : V \to W$ be a morphism of affine varieties and $\varphi^\sharp : \Gamma(W) \to \Gamma(V)$ the corresponding morphism of coordinate rings. Let $P \in V$ and $Q = \varphi(P)$ and consider local rings $\mathcal{O}_P(V)$, $\mathcal{O}_Q(W)$ with maximal ideals $\mathfrak{m}_P, \mathfrak{m}_Q$. Show that $\varphi^\sharp$ extends uniquely to a ring homomorphism $\mathcal{O}_Q(W) \to \mathcal{O}_P(V)$ and that $\varphi^\sharp(\mathfrak{m}_Q) \subseteq \mathfrak{m}_P$.

Solution 3. Note that we have the commutative diagram

$$\begin{array}{ccc} \Gamma(W) & \xrightarrow{\varphi^\sharp} & \Gamma(V) \\ & \searrow^{\mathrm{ev}_Q} & \downarrow{\scriptstyle \mathrm{ev}_P} \\ & & k \end{array}$$

and thus

$$(\varphi^\sharp)^{-1}(\mathfrak{m}_P) = (\varphi^\sharp)^{-1}(\mathrm{ev}_P^{-1}(0)) = (\mathrm{ev}_P \circ \varphi^\sharp)^{-1}(0) = \mathrm{ev}_Q^{-1}(0) = \mathfrak{m}_Q.$$

Note that by abuse of notation, we also denote the maximal ideals in $\Gamma(V)$ resp. $\Gamma(W)$ corresponding to P resp. Q by $\mathfrak{m}_P$ resp. $\mathfrak{m}_Q$.
Consider now the following commutative diagram

$$\begin{array}{ccc} \Gamma(W) & \xrightarrow{\varphi^\sharp} & \Gamma(V) \\ & {\scriptstyle \iota_P \circ \varphi^\sharp} \searrow & \downarrow {\scriptstyle \iota_P} \\ & & \Gamma(V)_{\mathfrak{m}_P} \end{array}$$

If now $f \in \Gamma(W) \setminus \mathfrak{m}_Q$, then as $(\varphi^\sharp)^{-1}(\mathfrak{m}_P) = \mathfrak{m}_Q$ we have $\varphi^\sharp(f) \notin \mathfrak{m}_P$, and thus $\iota_P \circ \varphi^\sharp(f)$ is a unit in $\Gamma(V)_{\mathfrak{m}_P}$. By the universal property of localization, there exists a unique map $\varphi^\sharp_P$ fitting in the commutative diagram

$$\begin{array}{ccc} \Gamma(W) & \xrightarrow{\varphi^\sharp} & \Gamma(V) \\ {\scriptstyle \iota_Q} \downarrow & & \downarrow {\scriptstyle \iota_P} \\ \Gamma(W)_{\mathfrak{m}_Q} & \xrightarrow[\varphi^\sharp_P]{} & \Gamma(V)_{\mathfrak{m}_P} \end{array}$$

Note that from this diagram, we infer that $\varphi^\sharp_P$ maps $f/g \in \Gamma(W)_{\mathfrak{m}_Q}$ to $\varphi^\sharp(f)/\varphi^\sharp(g) \in \Gamma(V)_{\mathfrak{m}_P}$. By point (3) of Proposition 3.11, we have $\Gamma(V)_{\mathfrak{m}_P} = \mathcal{O}_P(V)$ and $\Gamma(W)_{\mathfrak{m}_Q} = \mathcal{O}_Q(W)$, where a fraction f/g is mapped to $[D(g), f/g]$. It is then straightforward to see that the induced map $\varphi^\sharp_P\colon \mathcal{O}_Q(W) \to \mathcal{O}_P(V)$ maps $[U, f]$ to $[\varphi^{-1}(U), f \circ \varphi]$.

Remark.
- For a morphism of rings $\varphi\colon R \to S$ and maximal ideals $\mathfrak{m} \subseteq R$ and $\mathfrak{n} \subseteq S$, requiring $f(\mathfrak{m}) \subseteq \mathfrak{n}$ is equivalent to requiring $\mathfrak{m} \subseteq f^{-1}(\mathfrak{n})$. Clearly the latter implies the former. On the other hand, if we suppose that $f(\mathfrak{m}) \subseteq \mathfrak{n}$, then $f^{-1}(\mathfrak{n})$ is a non-trivial ideal of R containing $\mathfrak{m}$, so by maximality we must have $f^{-1}(\mathfrak{n}) = \mathfrak{m}$.
- Note that it is true for general varieties V, W (not necessarily affine), that a morphism $\varphi\colon V \to W$ induces a morphism of local rings $\varphi^\sharp_P\colon \mathcal{O}_Q(W) \to \mathcal{O}_P(V)$ for any $P \in V$ and $Q = \varphi(P) \in W$, and that furthermore $(\varphi^\sharp_P)^{-1}(\mathfrak{m}_P) = \mathfrak{m}_Q$. Indeed, we may define $\varphi^\sharp_P$ as

  $$\begin{aligned} \varphi^\sharp_P\colon \mathcal{O}_Q(W) &\to \mathcal{O}_P(V) \\ [U, f] &\mapsto [\varphi^{-1}(U), f \circ \varphi]. \end{aligned}$$

  Of course we need to check a couple of things: as φ is a morphism and $\varphi(P) = Q$, we have that $\varphi^{-1}(U)$ is an open set containing P and $f \circ \varphi$ is regular on $\varphi^{-1}(U)$. Furthermore, for two choices of representatives $[U, f] = [U', f']$, it is straightforward to check that $[\varphi^{-1}(U'), f' \circ \varphi]$, so $\varphi^\sharp_P$. Also, as precomposition with φ respects the ring operations, it is also straightforward to check that $\varphi^\sharp_P$ is a morphism of rings. At last, let us check that $(\varphi_P^\sharp)^{-1}(\mathfrak{m}_P) = \mathfrak{m}_Q$. For this, it suffices to see that we have the commutative diagram

$$\begin{array}{ccc} \mathcal{O}_Q(W) & \xrightarrow{\varphi_P^\sharp} & \mathcal{O}_P(V) \\ & \searrow^{\mathrm{ev}_Q} & \downarrow \mathrm{ev}_P \\ & & k \end{array}$$

In particular, we have

$(\varphi_P^\sharp)^{-1}(\mathfrak{m}_P) = (\varphi_P^\sharp)^{-1}(\mathrm{ev}_P^{-1}(0)) = (\mathrm{ev}_P \circ \varphi_P^\sharp)^{-1}(0) = \mathrm{ev}_Q^{-1}(0) = \mathfrak{m}_Q$.

which completes the construction.

- The construction is functorial: if we have morphisms $U \xrightarrow{\varphi} V \xrightarrow{\psi} W$, and $P \in U$, $Q = \varphi(P) \in V$ as well as $R = \psi(Q)$, then $(\psi \circ \varphi)_P^\sharp = \varphi_P^\sharp \circ \psi_Q^\sharp$.

Exercise 4. Let $n \geq 1$ and V a variety. We use projective coordinates $x_i,\ 1 \leq i \leq n+1$ on $\mathbb{P}_k^n$. Suppose there exist an open cover $(U_i)_{1 \leq i \leq n+1}$ of V and morphisms of varieties $\varphi_i : U_i \to \{x_i \neq 0\} \subseteq \mathbb{P}_k^n,\ 1 \leq i \leq n+1$, such that $\forall i \neq j,\ (\varphi_i)_{\restriction U_i \cap U_j} = (\varphi_j)_{\restriction U_i \cap U_j}$. Show that there exists a unique morphism $\varphi : V \to \mathbb{P}_k^n$ such that $\varphi_{\restriction U_i} = \varphi_i$. We say that φ is obtained by *glueing* the $\varphi_i,\ 1 \leq i \leq n+1$.

Solution 4. We can clearly define a map

$$\begin{aligned} \varphi \colon V &\to \mathbb{P}^n \\ x \in U_i &\mapsto \varphi_i(x) \end{aligned}$$

As the φ_i's agree on the intersections of the U_i's, it is well-defined. We need to show that it is a morphism. Let $f \in \mathcal{O}_{\mathbb{P}^n}(W)$ be a regular function on an open subset $W \subseteq \mathbb{P}^n$. Let $P \in \varphi^{-1}(W)$ be arbitrary and let i be such that $P \in U_i$. Then define $W_i \coloneqq W \cap \{x_i \neq 0\}$. Note that $f \circ \varphi = f \circ \varphi_i$ on $\varphi_i^{-1}(W_i)$, and $P \in \varphi_i^{-1}(W_i)$. As $f \circ \varphi_i$ is regular, we thus obtain that $f \circ \varphi$ is regular at P. As $P \in \varphi^{-1}(W)$ was arbitrary, we conclude that $f \circ \varphi$ is regular, i.e. $f \circ \varphi \in \mathcal{O}(\varphi^{-1}(W))$. Hence φ is a morphism.

Exercise 5. Let $f \in k[x_1, x_2, x_3]$ an irreducible form of degree 2 and consider $V_P(f) \subseteq \mathbb{P}_k^2$.

(1) Show that, up to a linear change of coordinates, we can assume that $f = x_2^2 - x_1 x_3$. (Hint: remember we classified similar subvarieties of $\mathbb{A}_k^2$).

(2) Show that the map:

$$\begin{array}{ccc} \mathbb{P}_k^1 & \to & \mathbb{P}_k^2 \\ (s:t) & \mapsto & (s^2 : st : t^2) \end{array}$$

induces an isomorphism $\mathbb{P}_k^1 \simeq V_P(f)$. (Hint: take a look locally in the standard affine opens of projective space and use exercise 4).

Solution 5.

(1) Let f_* be the dehomogenization of f (with respect to x_3). Note that $f_* \in k[x_1, x_2]$ is quadratic: indeed, we have $f = x_3^d (f_*)^*$ for some $d \geq 0$, and as f is irreducible we must have $d = 0$, i.e. $f = (f_*)^*$. As homogenization preserves the degree, we obtain that f_* is of degree 2. Furthermore, as $(\cdot)^*$ is multiplicative, we obtain that f_* is irreducible.

By (the solution of) Exercise 6 on Sheet 4, there exists a linear change of coordinates $T\colon \mathbb{A}^2 \to \mathbb{A}^2$ such that $f_* \circ T \in \{x_1x_2 - 1, x_1^2 - x_2\}$. Now let $T'\colon \mathbb{A}^3 \to \mathbb{A}^3$ the linear change of coordinates which is T on the first two components and the identity on the third. As $- \circ T'$ and $- \circ T$ preserve the degree of a polynomial, it is straightforward to check that

$$(g \circ T)^* = g^* \circ T'$$

for every $g \in k[x_1, x_2]$. In particular, we obtain

$$f \circ T' = (f_*)^* \circ T' = (f_* \circ T)^*,$$

so as $f_* \circ T \in \{x_1x_2 - 1, x_1^2 - x_2\}$, we obtain $f \circ T' \in \{x_1x_2 - x_3^2, x_1^2 - x_2x_3\}$. So by permuting the coordinates (which is linear), we obtain a linear change of coordinates T'' such that $f \circ T'' = x_2^3 - x_1x_3$.

(2) Related to Exercise 4 is the following slogan, which will be what we use: being a morphism of varieties is a local property. Concretely, what it means is the following:

Let V, W be varieties, and let $\varphi\colon V \to W$ be a map of sets. Suppose that there exists an open cover $W = \bigcup_i W_i$, and for every i a cover $\varphi^{-1}(W_i) = \bigcup_j V_{ij}$ with $V_{ij} \subseteq V$ open for every j, such that the (co-)restriction $\varphi|_{V_{ij}}^{W_i}$ is a morphism for all i, j. Then φ is a morphism.

Indeed, if $U \subseteq W$ is open, then

$$\varphi^{-1}(U) = \bigcup_i \varphi^{-1}(W_i \cap U) = \bigcup_i \bigcup_j V_{ij} \cap \varphi^{-1}(W_i \cap U) = \bigcup_{i,j} (\varphi|_{V_{ij}}^{W_i})^{-1}(W_i \cap U).$$

As $\varphi|_{V_{ij}}^{W_i}$ is continuous for all i, j, we obtain that $(\varphi|_{V_{ij}}^{W_i})^{-1}(W_i \cap U)$ is open in V_{ij}, and thus in V, for all i, j. Hence $\varphi^{-1}(U)$ is open and thus φ is continuous.

Now let $f \in \mathcal{O}(U)$ and $P \in \varphi^{-1}(U)$ be arbitrary. Let i be such that $\varphi(P) \in W_i$ and let j be such that $P \in V_{ij}$. Then on the open neighborhood $V_{ij} \cap \varphi^{-1}(W_i \cap U)$ of P, $f \circ \varphi$ agrees with $f \circ \varphi|_{V_{ij}}^{W_i}$, the latter of which is regular by hypothesis. Hence $f \circ \varphi$ is regular at every point of $\varphi^{-1}(U)$. So we conclude that φ is indeed a morphism.

Now to the exercise. Let $W_1 = V_p(f) \cap \{x_1 \neq 0\}$ and $W_2 = V_p(f) \cap \{x_3 \neq 0\}$, and note that $V_p(f) = W_1 \cup W_2$ is an open cover. Furthermore, $\varphi^{-1}(W_1) = \{s \neq 0\}$ and $\varphi^{-1}(W_2) = \{t \neq 0\}$ are open as well. Note that under the natural isomorphisms $\varphi^{-1}(W_i) = \mathbb{A}^1$ and $W_1 = V_a(f_{*1})$ resp.

$W_2 = V_a(f_{*3})$ (dehomogenization w.r.t the first resp. third variable), the (co-)restrictions $\varphi_i \coloneqq \varphi|_{\varphi^{-1}(W_i)}^{W_i}$ are given by

$$\begin{aligned} \varphi_1 \colon \mathbb{A}^1 &\to V_a(x_2^2 - x_3) \subseteq \mathbb{A}^2_{x_2,x_3} \\ t &\mapsto (t, t^2) \end{aligned}$$

(as $[1 : t]$ maps to $[1 : t : t^2]$), and

$$\begin{aligned} \varphi_2 \colon \mathbb{A}^1 &\to V_a(x_2^2 - x_1) \subseteq \mathbb{A}^2_{x_1,x_2} \\ s &\mapsto (s^2, s) \end{aligned}$$

(as $[s : 1]$ maps to $[s^2 : s : 1]$). These are clearly morphisms, so φ is a morphism.

In fact, φ_1 and φ_2 are isomorphisms, with inverse being projection to the first or second component. So we already obtain that φ is surjective.

Note also that φ is injective: let P be any point in the image of φ. If $P \in (W_1 \setminus W_2) \cup (W_2 \setminus W_1)$, then P has only one preimage as φ_i is an isomorphism. If $P \in W_1 \cap W_2$, we have to see that $\varphi_1^{-1}(P) = \varphi_2^{-1}(P)$. In this case, we have $P = [s^2 : st : t^2]$ with $s, t \neq 0$, and hence

$$\varphi_1^{-1}(P) = [1 : t/s] = [s/t : 1] = \varphi_2^{-1}(P).$$

Therefore, φ is injective.

In conclusion, we may consider the set-theoretic inverse $\psi = \varphi^{-1} \colon V_p(f) \to \mathbb{P}^1$. Then the (co-)restrictions $\psi_i \colon W_i \to \varphi^{-1}(W_i)$ satisfy $\psi_i = \varphi_i^{-1}$, so as φ_i is an isomorphism, ψ_i is in particular a morphism, and so ψ is a morphism, as being a morphism is a local property. Hence φ is an isomorphism.

In conclusion, we have $V_p(f) \cong \mathbb{P}^1$ for all irreducible forms $f \in k[x_1, x_2, x_3]$ of degree 2.